

# SLICING A 2-SPHERE

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**ABSTRACT.** We show that for every complete Riemannian surface  $M$  diffeomorphic to a sphere with  $k \geq 0$  holes there exists a Morse function  $f : M \rightarrow \mathbb{R}$ , which is constant on each connected component of the boundary of  $M$  and has fibers of length no more than  $52\sqrt{\text{Area}(M)} + \text{length}(\partial M)$ . We also show that on every 2-sphere there exists a simple closed curve of length  $\leq 26\sqrt{\text{Area}(S^2)}$  subdividing the sphere into two discs of area  $\geq \frac{1}{3}\text{Area}(S^2)$ .

## 1. INTRODUCTION

Let  $M$  be a Riemannian 2-sphere. Denote the area of  $M$  by  $|M|$ . In this paper we consider the problem of slicing  $M$  by short curves.

We start with the following isoperimetric problem: when is it possible to subdivide  $M$  into two regions of relatively large area by a short simple closed curve?

Papasoglu [9] used Besicovitch inequality to show that there exists a simple closed curve of length  $\leq 2\sqrt{3}|M| + \epsilon$  subdividing  $M$  into two regions of area  $\geq \frac{1}{4}|M|$ . A similar result was independently proved by Balacheff and Sabourau [3] using a variation of Gromov's filling argument.

On the other hand, consider the 3-legged starfish example on Figure 1.

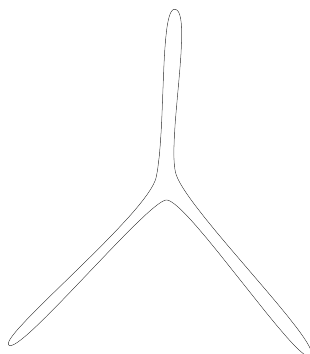


FIGURE 1. Example of a sphere that can not be subdivided into discs of approximately equal area by a short curve

As it has been observed in [11], for any  $r > \frac{1}{3}$ , if the tentacles are sufficiently thin and long, the length of the shortest simple closed curve subdividing  $M$  into two regions of area  $\geq r|M|$  can be arbitrarily large. A. Nabutovsky asked ([8]) the following question: what is the maximal value of  $r \in [\frac{1}{4}, \frac{1}{3}]$  such that for some  $c(r)$  each Riemannian 2-sphere of area 1 can be subdivided into two discs of area  $\geq r$  by a simple closed curve of length  $\leq c(r)$ ?

Our first result provides an answer for this question.

**Theorem 1.** *There exists a simple closed curve  $\gamma$  of length  $\leq 26\sqrt{|M|}$  subdividing  $M$  into two subdiscs of area  $\geq \frac{1}{3}|M|$ .*

To prove Theorem 1 we obtain the following result of independent interest.

**Theorem 2.** *There exists a map  $f$  from  $M$  into a trivalent tree  $T$ , such that fibers of  $f$  have length  $\leq 26\sqrt{|M|}$  and controlled topology: preimage of every interior point is a simple closed curve, preimage of every terminal vertex is a point and preimage of every vertex of degree 3 is homeomorphic to the greek letter  $\theta$ .*

Theorem 2 follows from a more general Theorem 10 in Section 3 for spheres with  $k \geq 0$  holes. Using different methods Guth [6] proved existence of a map from  $M$  into a trivalent tree with lengths of fibers bounded in terms of hypersphericity of  $M$ . It follows from Theorem 0.3 in [6] that there exists such a map with fibers of length  $\leq 34\sqrt{|M|}$ .

The second main result of this paper is about slicing  $M$  by 1-cycles. If instead of simple closed curves we allow subdivision by 1-cycles, we show that  $M$  can be subdivided into two regions with arbitrary prescribed ratio of areas by a 1-cycle of length  $\leq 52\sqrt{|M|}$ . In fact, we prove the following

**Theorem 3.** *There exists a Morse function  $f : M \rightarrow \mathbb{R}$  with fibers of length  $\leq 52\sqrt{|M|}$ .*

This improves the result of Balacheff and Sabourau [3] that there exists a sweep-out of  $M$  by 1-cycles of length  $\leq 10^8\sqrt{|M|}$ .

Theorem 3 follows from a more general result in Section 4, where we prove existence of a Morse function on a sphere with  $k$  holes, such that the function is constant on each connected component of the boundary and the length of fibers are bounded in terms of area and boundary length.

Alvarez Paiva, Balacheff and Tzanev [1] show that existence of a Morse function on a Riemannian 2-sphere with bounded fibers yields a length-area inequality for the shortest periodic geodesic on a Finsler 2-sphere (for both reversible and non-reversible metrics). Note that arguments used by Croke [4] to prove the length-area bound for the shortest closed geodesic on a Riemannian 2-sphere (see also [10] for

the best known constant) can not be directly generalized to the Finsler case because co-area inequality fails for non-reversible Finsler metrics.

Hence, Theorem 3 yields a better constant for Theorem VI in [1].

**Theorem 4.** *Let  $M$  be a Finsler two-sphere with Holmes-Thompson area  $A$ . Then  $M$  carries a closed geodesic of length  $\leq 160\sqrt{A}$ .*

The reason for constants 26 and 52 in our theorems is the following. We obtain the desired slicing of the sphere by repeatedly using the result of Papasoglu [9] to subdivide the sphere into smaller regions by a curve of length at most  $2\sqrt{3}$  times the square root of the area of the region. At each step the area of the region reduces at least by a factor of  $\frac{3}{4}$ . We then assemble these subdividing curves into one foliation with lengths bounded by the geometric progression

$$\sum_{i=0}^{\infty} 2\sqrt{3}\left(\frac{3}{4}\right)^{i/2} \sqrt{|M|} = 4\sqrt{3}(2 + \sqrt{3})\sqrt{|M|} \leq 26\sqrt{|M|}$$

In the proof of Theorem 3 some subdividing curves are used twice so an additional factor of 2 appears.

After the first subdivision happens our regions are no longer spheres, but rather spheres with a finite number of holes. It may not be possible to find a short simple closed curve subdividing it into two parts of area  $\leq \frac{3}{4}$  of its area. Instead we may have to use a collection of arcs with endpoints on the boundary subdividing the region into many pieces, each of small area. The issue is then how to assemble all of these subdividing curves into one foliation.

The main technical result of this paper (proved in the next section, see Proposition 7) is that we can always choose these subdividing arcs in such a way that they belong to a single connected component of the boundary of a certain subregion  $A_1$  with area of  $A_1$  between  $\frac{1}{4}$  and  $\frac{3}{4}$  of the area of the region. This result make assembling curves into one foliation a straightforward procedure.

After the first version of this article appeared on the web, F. Balacheff in [2] improved constant 26 in Theorems 1 and 2 to 7.6, and constant 160 in Theorem 4 to 31.1, but not constant 52 in Theorem 3.

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## 2. SUBDIVISION BY SHORT CURVES

Let  $M$  be a Riemannian 2-sphere. For  $0 < r \leq \frac{1}{2}$  let  $S_r(M)$  denote the set of simple closed curves on  $M$  that divide it into subdiscs of area  $\geq r|M|$ . Define  $c(M, r) = \inf_{\gamma \in S_r(M)} |\gamma|$  and  $c(r) = \sup c(M, r)$ , where the supremum is taken over all metrics on  $S^2$  of area 1.

By definition  $c(r)$  is increasing and by Proposition 2 below it is lower semicontinuous. For any  $r > \frac{1}{3}$  it follows from the example in the introduction (see Figure 1) that  $c(r) = \infty$ . For  $r = \frac{1}{4}$  we have the following result of Papasoglu.

**Theorem 5.** (*Papasoglu*, [9])  $c(\frac{1}{4}) \leq 2\sqrt{3}$

We will need to generalize Theorem 5 to spheres with finitely many holes and allow a larger class of subdividing curves than just simple closed curves. When the surface has boundary we will allow the subdividing curve  $\gamma$  to consist of several arcs with endpoints on the boundary. In this case we will define a distinguished connected component  $A_1$  of  $M \setminus \gamma$  and require that  $\gamma$  is contained in a connected component of  $\partial A_1$ . This is a technical condition that will make it easier to repeatedly cut the surface into smaller pieces and concatenate the subdividing curves to obtain a slicing of  $M$ .

Let  $M_k$  be a complete Riemannian 2-surface with boundary homeomorphic to a sphere with  $k$  holes. Let  $\gamma$  be a simple closed curve in the interior of  $M_k$  or a union of finitely many arcs  $\gamma = \bigcup \gamma_i$ , where  $\gamma_i$  are arcs with endpoints on  $\partial M_k$  that do not pairwise intersect and have no self-intersections. Let  $\{A_i\}$  be the set of connected components of  $M_k \setminus \gamma$ . Let  $S_{M_k}(r)$  denote the set of all such  $\gamma$  on  $M_k$  that in addition satisfy

- (1)  $r|M_k| \leq |A_1| \leq (1-r)|M_k|$
- (2)  $\gamma$  is contained in a connected component of  $\partial A_1$

In particular, (1) implies that the area of every connected component of  $M_k \setminus \gamma$  is bounded from above by  $(1-r)|M_k|$ .

Define  $c(M_k, r) = \inf_{\gamma \in S_{M_k}(r)} |\gamma|$  and  $c_k(r) = \sup c(M_k, r)$ , where the supremum is taken over all metrics on a sphere with  $k$  holes that have area 1.

We have the following useful fact.

**Proposition 6.**  $c_k(r)$  and  $c(r)$  are lower semi-continuous for  $r \in (0, \frac{1}{3}]$ .

*Proof.* We prove the result for  $c_k(r)$  and for  $c(r)$  it will follow as a special case from the argument below.

Let  $\{r_n\}$  be an increasing sequence converging to  $r$  for some  $0 < r \leq \frac{1}{3}$ . Fix  $\epsilon > 0$ . Let  $M_k$  be a complete Riemannian surface of area 1 diffeomorphic to a sphere with  $k$  holes. We would like to show that for some  $r_n$  there exists  $\gamma \in S_r(M_k)$  with  $|\gamma| \leq c_k(r_n) + \epsilon$ .

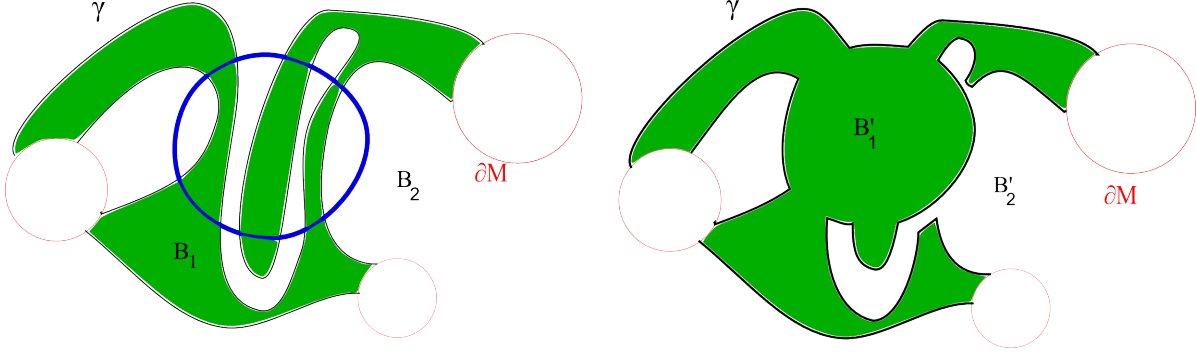


FIGURE 2. “Fattening” the smaller region

We can find  $\delta > 0$  small enough so that it satisfies the following requirements:

- (1) The area of the  $\delta$ -tubular neighbourhood of  $\partial M$  satisfies  $|N_\delta(\partial M_k)| < \frac{r}{100}$ .
- (2) Any ball  $B_\delta(x)$  around  $x \notin N_\delta(\partial M_k)$  is bilipschitz diffeomorphic to the Euclidean disc of radius  $\delta$  with Lipschitz constant between 0.99 and 1.01. In particular,  $3\delta^2 < |B_\delta(x)| < 4\delta^2$  and  $|\partial B_\delta(x)| < 8\delta$ .
- (3)  $\delta < \frac{\epsilon}{100}$

For such a  $\delta$  choose  $r_n$  so that  $r - r_n < (1 - r)\delta^2$ . Let  $\gamma \in S_{M_k}(r_n)$  be the subdividing arcs of length  $\leq c_k(r_n) + \epsilon/2$ . In the following argument it will be more convenient to consider  $\gamma'$ , the connected component of  $\partial A_1$  that contains  $\gamma$  (recall that by definition of  $c_k(r)$   $\gamma$  lies in a single boundary component of  $\partial A_1$ ). If  $\gamma$  is a closed curve then  $\gamma' = \gamma$ . If  $\gamma$  is a union of arcs then  $\gamma'$  is a closed curve made out of arcs of  $\gamma$  and arcs of  $\partial M$ .

Let  $B_1$  denote the element of  $\{A_1, M \setminus A_1\}$  of smaller area and  $B_2$  denote the element of larger area. We can assume that  $|B_1| < r$  for otherwise we are done. Therefore, we have  $r_n \leq |B_1| < r$  and  $1 - r < |B_2| \leq 1 - r_n$ .

Let  $M'$  denote  $M \setminus N_\delta(\partial M_k)$ . By our choice of  $\delta$  we have that the area of a ball  $|B_\delta(x)| \geq 3\delta^2$  for  $x \in M'$  and  $|M' \cap B_2| \geq 0.99|B_2|$ . By Fubini's theorem we obtain  $\int_{M'} |B_\delta(x) \cap B_2| = |B_2 \cap M'| \int_{M'} |B_\delta(x)| \geq 3/2(1 - r)|M'|\delta^2$ . Hence, for some  $x \in M'$  we have  $|B_\delta(x) \cap B_2| \geq 3/2(1 - r)\delta^2$ . Since  $M' \cap B_1$  is non-empty we can always find such a ball so that  $\gamma' \cap B_\delta(x)$  is non-empty.

We will now construct a new curve  $\beta$  that coincides with  $\gamma'$  outside of  $B_{1.1\delta}(x)$  and divides  $M_k$  into regions  $B'_1$  and  $B'_2$  so that one of them is connected and each of them has area  $\geq r$ . Moreover,  $|\beta| < |\gamma'| + \epsilon/2$ . This implies the desired inequality  $c(r, M_k) \leq c(r_n, M_k) + \epsilon$ .

We construct  $\beta$  by cutting  $\gamma'$  at the points of intersection with  $\partial B_\delta(x)$  and attaching arcs of  $\partial B_\delta(x)$ . We do it in such a way that  $B_\delta(x)$  is now entirely contained in the smaller of two regions. This increases the area of the smaller region

by at least  $(1 - r)\delta^2$  and increases the length of the subdividing curve by at most  $2|\partial B_\delta(x)| \leq 16\delta < \epsilon/2$ .

The procedure is illustrated in Figure 2. Let  $\{C_j\}$  be connected components of  $B_1 \setminus B_\delta(x)$  and  $\{c_i^j\}$  denote connected components of  $\partial C_j \cap \partial B_\delta(x)$ . First we erase all arcs of  $\gamma'$  that are in  $B_\delta(x)$ . For each  $j$  we erase the arc  $c_1^j$  from  $\partial B_\delta(x)$ . For each  $c_i^j$ ,  $i > 1$ , we add a copy of  $c_i^j$  and perturb it so that the new curve  $\beta$  does not intersect  $B_\delta(x)$  in the neighbourhood of  $c_i^j$ . This does not increase the number of connected components of either region.  $\square$

The only example that I know where upper semicontinuity of  $c(r)$  fails is the three-legged starfish on Figure 1 (showing  $c(r) = \infty$  for  $r > \frac{1}{3}$ , while  $c(1/3) \leq 26$  by Theorem 1). The following question seems natural:

**Question** Is  $c(r)$  continuous for  $r \in (0, \frac{1}{3})$ ?

It can be easily shown from the definition that  $c(r) \leq c_k(r)$ . Under the additional assumption  $r \leq \frac{1}{4}$  we are able to prove that they are equal.

**Proposition 7.** *Suppose  $r \leq \frac{1}{4}$ , then  $c_k(r) = c(r)$ .*

In the proof of Proposition 7 we will use the following topological fact.

**Lemma 8.** *Let  $M$  be a submanifold (with boundary) of  $S^2$  and let  $\gamma$  be a simple closed curve in  $S^2$ . Suppose the intersection of  $\gamma$  and  $\partial M$  is non-empty and transversal. If  $A$  denotes a connected component of  $S^2 \setminus \gamma$  then there exists an arc  $a \subset \gamma$  and an arc  $b \subset \partial M$ , such that  $a \cup b$  bounds a disc  $D_{a \cup b} \subset A$  and  $D_{a \cup b} \cap M$  is connected.*

The proof of this Lemma is the main technical part of the paper. The proof is illustrated on Figure 3. The reader can draw several pictures like on Figure 3 and convince him or herself that the statement is correct.

*Proof.* The strategy of the proof is to find some disc  $D_{a' \cup b'}$ , with possibly a large number of components of  $D_{a' \cup b'} \cap M$ . We then show that there exists a subdisc of  $D_{a' \cup b'}$  whose intersection with  $M$  has a smaller number of connected components.

Let  $p_1 \in \partial M \cap \gamma$  and  $C_1$  be the connected component of  $\partial M$  that contains  $p_1$ .

Define an interval  $b_1 \subset C_1$  to be a connected component of  $C_1 \cap A$  that has  $p_1$  as an endpoint (note that it is unique). We denote the other endpoint of  $b_1$  by  $q_1$ . Let  $v$  be a tangent vector to  $\gamma$  at  $p_1$  pointing inside  $M$ . Let  $a_1$  be an arc of  $\gamma$  that starts at  $p_1$  in the direction  $v$  and ends at  $q_1$ .

Observe that the curve  $a_1 \cup b_1$  is a simple closed curve enclosing a disc  $D_{a_1 \cup b_1} \subset A$  that contains a non-empty subset of  $M \cap A$ . If  $\bar{A}_1 = D_{a_1 \cup b_1} \cap M$  has one connected component then we are done. Assume it has more than one. Let  $A_1$  be a component

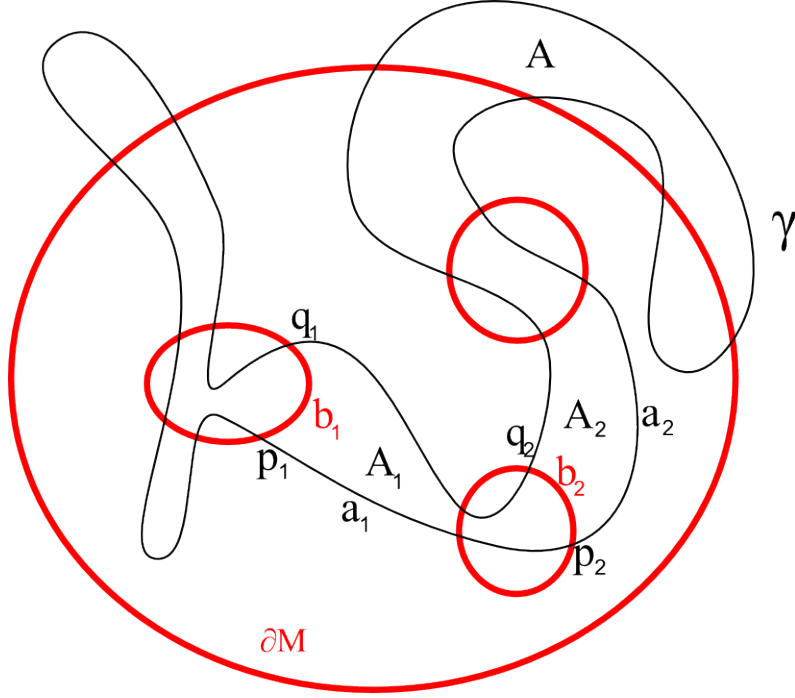


FIGURE 3. Proof of Lemma 8

of  $\overline{A}_1$  with  $b_1 \subset \partial A_1$  and let  $A_2$  be a different component. Define a point  $p_2 \in a_1$  by

$$p_2 = a_1\left(\inf_{0 \leq t \leq 1} \{t | a_1(t) \in \partial A_2\}\right)$$

It follows from the definition that  $p_2 \in \partial M$ . We can find a point  $q_2 \in \gamma \cap \partial M$  and an arc  $b_2 \subset \partial M \cap A$  from  $p_2$  to  $q_2$ .

The interior of  $\gamma \setminus a_1$  is contained in the interior of  $S^2 \setminus D_{a_1 \cup b_1}$ . It follows that  $q_2 \in a_1$ . Denote the arc of  $a_1$  between  $p_2$  and  $q_2$  by  $a_2$ . We have that  $a_2 \cup b_2$  separates  $S^2$  into a disc  $D_{a_2 \cup b_2}$  that contains  $A_2$  and its complement that contains  $A_1$ . Set  $\overline{A}_2 = D_{a_2 \cup b_2} \cap A$ . It is non-empty and has fewer connected components than  $\overline{A}_1$ . We iterate this procedure until we are left with just one connected component.  $\square$

We now prove Proposition 7.

The direction  $c(r) \leq c_k(r)$  is simple. Given a Riemannian 2-sphere we can make  $k$  holes in it of small area and small boundary length. We subdivide the resulting sphere with holes and use the fact that the connected component of  $\partial A_1$  that contains  $\gamma$  will have length close to  $c_k(r)$ . Here we did not use that  $r \leq \frac{1}{4}$ .

To prove the other direction we proceed as follows. Let  $\epsilon$  be a small positive constant. Given a sphere with  $k$  holes  $M_k$  of area 1 we attach  $k$  discs of total area  $\epsilon$  to the boundary of  $M_k$ . We obtain a Riemannian 2-sphere  $M$ .

Let  $\gamma_1$  be a simple closed curve of length  $\leq c_r(M) + \epsilon$  subdividing  $M$  into two subdiscs  $A$  and  $B$  of area between  $r(1+\epsilon)$  and  $(1-r)(1+\epsilon)$ . Suppose  $A$  is a subdisc of area  $\geq \frac{1}{2}(1+\epsilon)$ . If  $\gamma_1$  is disjoint from  $\partial M_k$  we conclude that  $c(M_k, \frac{r}{1+\epsilon}) \leq c_r(M) + \epsilon$ .

Suppose  $\gamma_1$  intersects  $\partial M_k$ . By Lemma 8 there exists an arc  $a_1$  of  $\gamma_1$  and an arc  $b_1$  of  $\partial M_k$ , such that  $A_1 = D_{a_1 \cup b_1} \cap M_k \subset A$  is connected.

We consider two possibilities. First, suppose  $|A_1| \geq r(1+\epsilon)$ . Let  $\gamma_2$  be the intersection of  $a_1$  with the interior of  $M_k$ .  $A_1$  is a connected component of  $M_k \setminus \gamma_2$  of area between  $r - \epsilon$  and  $1 - r + \epsilon$ . The rest of connected components of  $M_k \setminus \gamma_2$  have area less than  $1 - r + \epsilon$ . So  $c(M_k, r - \epsilon) \leq c_r(M) + \epsilon$ .

Alternatively, suppose  $|A_1| < r(1+\epsilon)$ . In this case we can define a new curve  $\gamma_2$ , such that the number of connected components of  $M_k \setminus \gamma_2$  is smaller than the number of connected components of  $M_k \setminus \gamma_1$ . We do this by replacing the arc  $a_1$  of  $\gamma_1$  by  $b_1 \subset \partial M_k$ . Note that we can slightly perturb the part of the new curve that coincides with  $b_1$  so that it is entirely in  $M \setminus M_k$ , in particular, the intersection of  $\gamma_2$  with  $\partial M_k$  is transversal and the length of the intersection of  $\gamma_2$  with the interior of  $M_k$  is smaller than that of  $\gamma_1$ . As a result we transferred the area of  $A_1$  from  $A$  to  $B$ . Since  $|A_1| < r(1+\epsilon)$ ,  $r \leq \frac{1}{4}$  and  $|A| \geq \frac{1+\epsilon}{2}$  we obtain that the area of each of the two discs  $M \setminus \gamma_2$  is at least  $r(1+\epsilon)$ . In this way we can continue reducing the number of connected components of  $M_k \setminus \gamma_i$  until one of the subdiscs of  $M \setminus \gamma_i$  contains only one connected component of  $M_k$  or until we encounter the first possibility above.

By Proposition 6 we conclude that  $c_k(r) \leq c(r)$ .

### 3. $T$ -MAPS

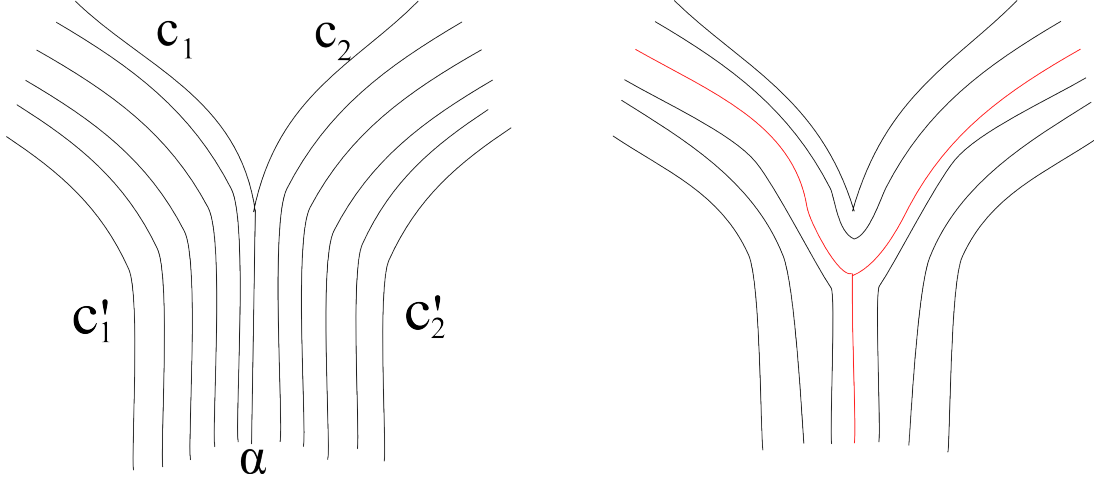
**Definition 9.** A map  $f$  from  $M_k$  to a trivalent tree  $T$  is called a  $T$ -map if the topology of fibers of  $f$  is controlled in the following way: the preimage of any point in an edge of  $T$  is a circle, there exist  $k$  terminal vertices  $x_k \in T$ , such that  $f^{-1}(x_k)$  is a connected component of  $\partial M_k$ , the preimage of any other terminal point of  $T$  is a point, and the preimage of a trivalent vertex of  $T$  is homeomorphic to the greek letter  $\theta$ .

**Theorem 10.** For  $r \in (0, \frac{1}{4}]$  and any  $\epsilon > 0$  there exists a  $T$ -map  $f$  from  $M_p$ ,  $p \geq 0$ , so that each fiber of the map has length less than  $\frac{c(r)}{1-\sqrt{1-r}} + |\partial M_p| + \epsilon$ .

Theorem 2 follows by taking  $r = \frac{1}{4}$  and applying Theorem 5.

In the proof we will repeatedly use the following simple fact, so it is convenient to state it as a separate lemma.



FIGURE 4. Surgery in the neighbourhood of  $a_i$ 

**Lemma 11.** *Let  $A_1$  and  $A_2$  be two closed smooth submanifolds (with boundary) of  $M_p$ , such that  $\alpha = A_1 \cap A_2$  is a connected arc. Let  $c_i$  denote the connected component of  $\partial A_i$  that contains  $\alpha$ . Suppose  $|c_1 \cup c_2| < L$  and that each  $A_i$  admits a  $T$ -map with fibers of length  $< L$ , then  $A_1 \cup A_2$  admits a  $T$ -map with fibers of length  $\leq L$ .*

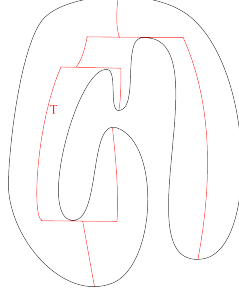
*Proof.* From the assumption that  $A_i$  admits a  $T$ -map it follows that there exists an embedded cylinder  $C_i \subset A_i$  and a map  $f_i : C_i \rightarrow [0, 1]$  with fiber  $f_i^{-1}(0) = c_i$  and the length of all fibers  $< L$ . The boundary  $\partial C_i = c_i \cup c'_i$  with  $c'_i$  contained in the interior of  $A_i$ .

Let  $a_1$  and  $a_2$  be the endpoints of  $\alpha$ . For a sufficiently small  $\delta > 0$  we perform a surgery on the the closed curves in the foliation  $\{f_i^{-1}(t) | 0 \leq t \leq \delta\}$ . The surgery happens in the  $\delta$ -neighbourhood of  $a_i$  and is depicted on Figure 4.

As a result of the surgery we obtain three families of curves. The “outer” family converging to  $c_1 \cup c_2$ , and two “inner” families each converging to  $c'_1$  or  $c'_2$ . These three families are separated by a  $\theta$  graph (drawn in red on Figure 4). This surgery defines the desired  $T$ -map. □

We will need first a version of Theorem 10 for very small balls.

**Lemma 12.** *For any  $\epsilon > 0$  there exists  $l > 0$ , such that for every disc  $D \subset M_p$  with  $|\partial D| \leq l$  there exists a diffeomorphism  $f$  from  $D$  to the standard closed disc  $D_{st} = \{x^2 + y^2 \leq 1\}$  so that the preimage of each concentric circle  $f^{-1}(\{x^2 + y^2 = \text{const}\})$ , has length  $\leq (1 + \epsilon)|\partial D|$ .*

FIGURE 5. Tree  $T$  in  $D'$ 

*Proof.* For  $\delta > 0$  sufficiently small every ball  $B \subset M_p$  of radius  $\delta$  is  $(1+\epsilon)$ -bilipschitz diffeomorphic to a disc in the closed upper half-plane  $\mathbb{R}_+^2$ .

Let  $D'$  denote the image of  $D$  under such a diffeomorphism. (Here we are assuming that  $l < \delta$  and  $D$  is contained within a ball of radius  $\delta$ ). After a small perturbation we may assume that the projection  $p$  of  $\partial D'$  onto  $y$  coordinate is a Morse function.

Define a 1-dimensional simplicial complex  $T \subset D'$  as follows. Let  $a$  be a regular value of the projection function  $p|_{\partial D'}$  restricted to the boundary of  $D'$ .  $p^{-1}(a) \cap D'$  is a finite union of disjoint closed intervals  $\{v_i\}$ . We set  $T \cap p^{-1}(a)$  to be the midpoints of  $v_i$ 's. If  $a$  and  $b$  are two consecutive critical values of  $p$ , it follows that  $T \cap p^{-1}(a, b)$  is a collection of disjoint simple arcs as on Figure 5.

At a critical point  $\partial D'$  locally looks like the graph of a function  $f(x) = \pm x^2$ . We connect the endpoints of the intervals of  $T$  by a horizontal arc tangent to the critical point and contained in  $D'$ .

Note that  $D'$  retracts onto  $T$ , so in particular  $T$  must be connected and simply connected, hence a tree. We contract  $D$  along the edges of  $T$  in the obvious way. As a result we obtain a contraction of  $\partial D$  inside  $D$  to a point through curves of length  $\leq (1+\epsilon)|\partial D|$ . After a small perturbation we can assume that this homotopy realizes the desired diffeomorphism. For details we refer the reader to [5], where it is shown that if there exists a homotopy of the boundary of  $D$  to a point through curves of length  $< L$ , then there exists a diffeomorphism from  $D$  to  $D_{st}$  so that preimages of concentric circles have length  $< L$ .

□

**Lemma 13.** *For any  $\epsilon > 0$  there exists  $A > 0$ , such that for every disc  $D \subset M_p$  with  $|D| \leq A$  there exists a  $T$ -map  $f$  from  $D$  with fibers of length less than  $|\partial D| + \epsilon$ .*

*Proof.* The proof is similar to that of Lemma 2.2 in [7].

Choose  $A < \min\{\epsilon, \frac{l^2}{64}\}$ , where  $l$  is as in Lemma 12. The proof is by induction on  $n = \lceil \frac{|\partial D|}{\sqrt{|A|}} \rceil$ . For  $n \leq 8$  the result follows by Lemma 12. Assume the Lemma to be

true for all subdiscs with  $\lceil \frac{|\partial D|}{l} \rceil < n$  and consider the case when this quantity equals  $n \geq 8$ .

Subdivide  $\partial D$  into 4 arcs of equal length. By Besicovitch Lemma we can find an arc  $\alpha$  of length  $\leq \sqrt{|A|}$  connecting two opposite arcs.  $\alpha$  subdivides  $D$  into two subdiscs  $D_1$  and  $D_2$  of area  $\leq A$  and boundary length  $\leq \frac{3}{4}|\partial D| + \sqrt{|A|} \leq (n-1)\sqrt{|A|}$ . Hence, by inductive assumption  $D_1$  and  $D_2$  admit  $T$ -maps with fibers of length  $\leq |\partial D|$ .

By inductive assumption each disc admits a  $T$ -map with fibers of length  $\leq |\partial D| + \epsilon$ . We also have  $|\partial D_1 \cup \partial D_2| \leq |\partial D| + \epsilon$  so the result follows by Lemma 11.  $\square$

We need to generalize this result about small discs to other small submanifolds of  $M_p$ .

**Lemma 14.** *For any  $\epsilon > 0$  there exists  $A > 0$ , such that for every submanifold with boundary  $M_k \subset M_p$ , with  $|M_k| \leq A$  there exists a  $T$ -map  $f$  from  $M_k$  with fibers of length  $\leq |\partial M_k| + 4(k-1)\sqrt{A} + \epsilon$ .*

*Proof.* Let  $M_k$  be a closed submanifold (with boundary) of  $M_p$  and let  $c$  be a connected component of  $\partial M_k$ . If  $\text{dist}(c, \partial M_k \setminus c) = d$  then there exists an open ball  $B(d/2) \subset M_k$  of radius  $d/2$  whose interior does not intersect the boundary of  $M_k$  (and, in particular, it does not intersect the boundary of  $M_p$ ). As  $d \rightarrow 0$  the area of  $B(d/2)$  approaches the area of a Euclidean disc of the same diameter. Since  $M_p$  is compact, this happens uniformly for all balls of radius  $d/2$  disjoint from the boundary. For  $d$  sufficiently small we may conclude that  $|M_k| \geq 3(d/2)^2$ . Hence, for a sufficiently small  $A$ , if  $|M_k| \leq A$  then the distance  $\text{dist}(c, \partial M_k \setminus c) \leq \frac{2\sqrt{3}}{3}\sqrt{A}$ . We attach  $k-1$  arcs  $\{\gamma_i\}$  to the boundary of  $\partial M_k$  of total length less than  $2(k-1)\sqrt{A}$  and so that  $\partial M_k \cup \bigcup \gamma_i$  is connected and its complement in  $M_k$  is homeomorphic to a disc. Denote this disc by  $D$ .

Consider the normal  $\delta$ -neighbourhood  $N_\delta$  of  $\partial D$  in  $M_k$  (see Figure 6) for some small  $\delta$ . By Lemma 13 the complement of  $N_\delta$  in  $M_k$  admits a  $T$  map with fibers of length  $\leq |\partial M_k| + 4(k-1)\sqrt{A} + \epsilon$ . Let  $\alpha$  be a short closed curve in  $N_\delta$  which separates one connected component of  $\partial M_k$  from other connected components as on Figure 6. Curve  $\alpha$  separates  $N_\delta$  into two regions. Let  $B$  denote the region that contains only one connected component of  $\partial M_k$ . By Lemma 11 we can extend the  $T$ -map to  $D \cup B$ . By chopping off connected components of  $\partial M_k$  and applying Lemma 11 repeatedly we obtain the desired  $T$ -map.  $\square$

The proof of Theorem 10 proceeds inductively by cutting  $M$  into smaller pieces until their size is small enough so that Lemma 14 can be applied. We assemble

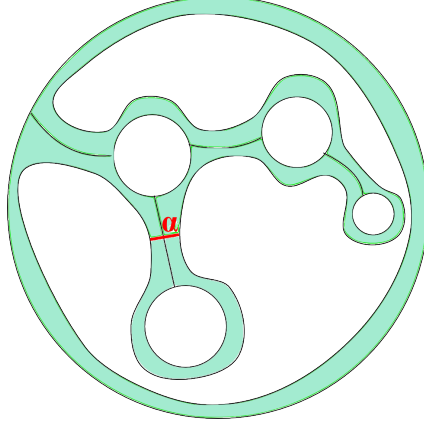


FIGURE 6. Inductive step of the proof of Lemma 14

$T$ -maps on these smaller regions to obtain one map from  $M$  with the desired bound on lengths of fibers.

Fix  $\epsilon > 0$  and let  $A$  be as in Lemma 14. Let  $N = \lceil \log_{4/3}(\frac{|M|}{A}) \rceil$ . We claim that for every  $M_k \subset M_p$  with  $(\frac{3}{4})^{n+1}|M_p| < |M_k| \leq (\frac{3}{4})^n|M_p|$  there exists a  $T$ -map with fibers of length

$$(1) \quad \leq |\partial M_k| + 4(N - n + k)\sqrt{A} + \sum_{i=n}^{N-1} c(r)\left(\frac{3}{4}\right)^{i/2} \sqrt{|M_k|} + \epsilon$$

When  $n = N$  the inequality (1) is true by Lemma 14. We assume it to be true for  $n + 1 \leq N$  and prove it for  $n$ .

By Proposition 7 there exists  $\gamma \in S(M_k, r)$  of length  $\leq c(r) + \epsilon'$ .

We have two possibilities.

**Case 1.**  $\gamma$  is a simple closed curve.

In this case  $\gamma$  separates  $M_k$  into two regions  $N_1$  and  $N_2$  of area  $|N_i| \leq (\frac{3}{4})^{n+1}|M|$ . The number of connected components of  $\partial N_i$  is at most  $k + 1$ .

By inductive assumption  $N_1$  and  $N_2$  admit  $T$ -maps into trees  $T_1$  and  $T_2$  respectively. We construct  $T$  by identifying the terminal vertex  $f_1(\gamma)$  of  $T_1$  with the terminal vertex  $f_2(\gamma)$  of  $T_2$ .  $T$ -map  $f$  is defined by setting it equal to  $f_i$  when restricted to  $N_i$ . A simple calculation shows that lengths of fibers of  $f$  satisfy the desired bound.

**Case 2.**  $\gamma$  is a collection of arcs with endpoints on  $\partial M_k$ .

Let  $\{S_1, \dots, S_j\}$  be connected components of  $\partial M_k$  that intersect arcs of  $\gamma$ . For a sufficiently small  $\delta$  the normal neighbourhood  $N_\delta(S_i)$  is foliated by closed curves of length very close to  $|S_i|$ , which are transverse to the arcs of  $\gamma$ . In particular, each  $N_\delta(S_i)$  admits a  $T$ -map with fibers of length  $\leq |S_i| + \epsilon'$ . Let  $A_1$  be the connected

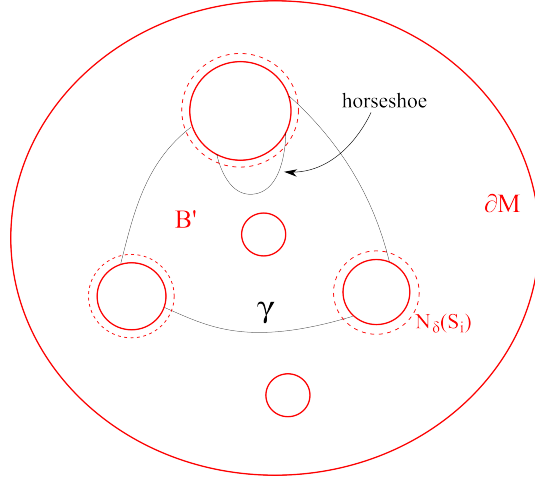


FIGURE 7. Case 2 in the proof of Theorem 10

component of  $M \setminus \gamma$  with area satisfying  $r|M| \leq |A_1| \leq (1-r)|M|$  (recall that it exists by definition of  $c_k(r)$ ). Let  $B$  denote  $A_1 \setminus \bigcup N_\delta(S_i)$  and  $M'_k$  denote  $M_k \setminus N_\delta(S_i)$ . (See Figure 7).

Let  $C$  be the connected component of  $\partial B$  that contains  $\gamma \cap M'_k$ . We say that an arc  $\alpha$  of  $\gamma$  is a horseshoe if  $\alpha$  is in  $M'_k$  and the endpoints of  $\alpha$  lie on the same connected component of  $\partial M'_k$ . We will use the following simple observation.

**Lemma 15.** *If  $\gamma$  contains no horseshoes, then there exists a connected component  $S$  of  $\partial M'_k$ , such that  $S \cap C$  is connected.*

*Proof.* Let  $S$  be a connected component of  $\partial M'_k$  and suppose  $S \cap C$  contains more than one interval. Let  $S'$  be an interval of  $S \setminus C$  and let  $a$  and  $b$  denote the endpoints of  $S'$ .  $C \setminus a \cup b$  consists of two arcs, call them  $C_1$  and  $C_2$ . Let  $S_1 \neq S$  be a connected component of  $\partial M'_k$  that intersects  $C_1$ . We claim that  $C_2$  does not intersect  $S_1$ . For suppose it does, consider then a subarc  $C'_1$  of  $C_1$  from  $a$  until the first point of intersection with  $S_1$  and denote this point by  $a_1$ . Similarly, denote by  $C'_2$  the subarc of  $C_2$  from  $a$  until the first point of intersection with  $S_1$  and call it  $a_2$ . Let  $S'_1$  be an arc of  $S_1$  from  $a_1$  to  $a_2$ , then  $\beta = C'_1 \cup C'_2 \cup S'_1$  is a closed curve separating  $M'_k$  into two connected components. Moreover, the point  $b$  and points of  $(C'_1 \cup C'_2) \cap S$  belong to different connected components of  $M'_k \setminus \beta$ . This is a contradiction since  $B$  is connected.

Suppose the intersection  $C \cap S_1$  is not connected. We can then find a subarc  $C_3 \subset C_1$  that intersects  $S_1$  only at the endpoints. The number of connected components of  $\partial M'_k$  that intersect  $C_3$  is strictly smaller than the number of components that intersect  $C_1$ . Proceeding in this way we can find an arc of  $C$  that has endpoints on

$S_j$ , and its interior intersects only one connected component  $S_{j+1}$  of  $\partial M'_k$ . It follows that  $S_{j+1} \cap C$  is connected.  $\square$

We can now construct the desired  $T$ -map. By inductive assumption there exists a  $T$ -map on  $B$  with fibers  $\leq |\partial B| + 4(N - n - 1 + k')\sqrt{A} + \sum_{i=n+1}^{N-1} c(r)(\frac{3}{4})^{i/2}\sqrt{|M|} + \epsilon'$ . Note that  $|\partial B| \leq |\partial M_k| + c(r)\sqrt{|M_k|} + \epsilon'$  and the number of connected components  $k'$  of  $\partial B$  satisfies  $k' \leq k$ .

Suppose first that  $\alpha \subset \partial B$  is a horseshoe. It separates  $M'_k$  into two connected components. Let  $B_2$  denote the connected component that does not contain  $B$ . Since  $|B_2| \leq (\frac{3}{4})^{n+1}|M|$ , it admits a  $T$ -map with the desired bound on length of fibers. By Lemma 11 we can extend the  $T$ -map to  $B \cup B_2$ . Inductively we extend the  $T$ -map to a subset  $B' \subset M'_k$ , such that  $B \subset B'$  and  $\partial B'$  does not contain any horseshoes. By Lemma 15  $\partial B' \cap N_\delta(S_i)$  is connected for some  $i$ . By Lemma 11 we can extend the  $T$ -map to  $B' \cup N_\delta(S_i)$ . We iterate this procedure until we extended the  $T$ -map to a set  $B''$ , which contains  $N_\delta(S_i)$  for all  $i$ . The last step is to extend the  $T$ -map to the whole of  $M_k$  exactly as we did it in Case 1.

To finish the proof of Theorem 10 recall that  $N\sqrt{A} = \lceil \log_{4/3}(\frac{|M|}{A}) \rceil \sqrt{A} \rightarrow 0$  as  $A \rightarrow 0$ . Since  $A$  can be chosen arbitrarily small Theorem 10 follows from (1).

Now we can prove Theorem 1. Let  $M$  be a Riemannian 2-sphere and  $f : M \rightarrow T$  a  $T$ -map with fibers of length  $\leq 26\sqrt{|M|}$ . If there are no trivalent vertices in  $T$  we can find a short closed curve that separates  $M$  into two halves of equal area.

For each trivalent vertex  $v_k \in T$  let  $\alpha_1^k \cup \alpha_2^k \cup \alpha_3^k = \theta_k = f^{-1}(v_k)$ . Denote the closed curve  $\alpha_i^k \cup \alpha_j^k$  by  $\gamma_{ij}^k$ . Assume that for every  $k, i, j$  ( $i \neq j$ )  $\gamma_{ij}^k$  separates  $M$  into two discs, s.t. the area of the smaller disc is strictly smaller than  $\frac{1}{3}|M|$ . Let  $k, i, j$  be such that the area of the smaller disc is maximized among all such curves.

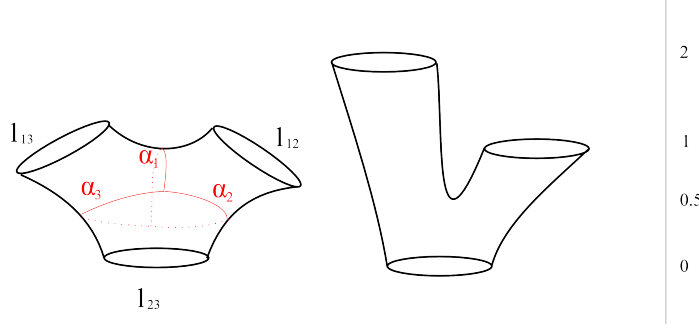
Let  $D_s$  denote the smaller and  $D_l$  denote the larger of the subdiscs of  $M \setminus \gamma_{ij}^k$ . Let  $e$  denote the edge of  $T$  adjacent to  $v_k$ , such that  $f^{-1}(e) \subset D_l$ . We observe that the other two edges adjacent to  $v_k$  are contained in  $f(D_s)$ , for otherwise it would follow that the area of the smaller disc is not maximal for  $\gamma_{ij}^k$ .

We conclude that for some  $x \in e$   $f^{-1}(x)$  subdivides  $M$  into two discs of area  $\geq \frac{1}{3}|M|$  for otherwise it would again contradict our choice of  $\gamma_{ij}^k$ .

#### 4. MORSE FUNCTION

In this section we prove the existence of a Morse function  $f$  from  $M$  to  $\mathbb{R}$  with short preimages.

**Definition 16.** *Let  $M$  be a manifold with boundary. A function  $f : M \rightarrow \mathbb{R}$  is called an  $m$ -function if it is Morse on the interior of  $M$ , constant on each boundary component and maps boundary components to distinct points disjoint from the critical values of  $f$ .*

FIGURE 8. Constructing an  $m$ -function on the neighbourhood of  $\theta$ .

**Theorem 17.** For  $r \in (0, \frac{1}{4}]$  and any  $\epsilon > 0$  there exists an  $m$ -map  $f$  from  $M_p$ ,  $p \geq 0$ , so that each fiber of the map has length less than  $\frac{2c(r)}{1-\sqrt{1-r}} + |\partial M_p| + \epsilon$ .

Theorem 3 follows by taking  $r = \frac{1}{4}$  and applying Theorem 5.

The proof of Theorem 17 proceeds along the same lines as the proof of Theorem 10. There are two main differences. The first difference is that we would like the function to be smooth (in particular, curves in the foliation  $f^{-1}(x)$  are not allowed to have corners) and have singularities of Morse type. This is accomplished by a simple surgery described in Lemma 18.

The second difference is that we would like to bound the length of the whole level set, not just of the individual connected components.

We will need one technical lemma similar to concatenation Lemma 11. Let  $\theta \subset M_p$  denote a union of three non-intersecting simple arcs  $\alpha_i$  with common endpoints.  $\theta$  subdivides  $M_p$  into three regions  $U_{12}, U_{23}, U_{13}$ , choosing the indices so that  $\alpha_i \cup \alpha_j \subset \partial U_{ij}$ . Let  $l_{ij}$  denote a curve obtained by pushing  $\alpha_i \cup \alpha_j$  inside  $U_{ij}$  by a small perturbation, so that it is contained in a small normal neighbourhood of  $\theta$ , smooth and has length  $< |\alpha_i| + |\alpha_j| + \epsilon$ . Let  $U(\theta)$  denote the neighbourhood of  $\theta$  bounded by curves  $l_{ij}$  (see Figure 8).

**Lemma 18.**  $U(\theta)$  admits an  $m$ -map  $f : U(\theta) \rightarrow [0, 2]$  with  $f(l_{23}) = 0$ ,  $f(l_{12}) = 1$ ,  $f(l_{13}) = 2$ , and fibers satisfying the following inequalities:

- (1) For  $x \in [0, 0.5]$ ,  $|f^{-1}(x)| \leq |\alpha_2| + |\alpha_3| + \epsilon$
- (2) For  $x \in (0.5, 1]$ ,  $|f^{-1}(x)| \leq 2|\alpha_1| + |\alpha_2| + |\alpha_3| + \epsilon$
- (3) For  $x \in (1, 2]$ ,  $|f^{-1}(x)| \leq |\alpha_1| + |\alpha_3| + \epsilon$

Finally, let  $n_{ij}$  denote an outward unit normal at some point on  $l_{ij}$ , then  $df(n_{23}) < 0$ ,  $df(n_{12}) > 0$  and  $df(n_{13}) > 0$ .

*Proof.* Let  $a$  and  $b$  denote the endpoints of  $\alpha_i$ . Since  $M$  is orientable, the normal tubular neighbourhood of  $\alpha_i \cup \alpha_j$  is homeomorphic to the cylinder. Let  $l_{ij}$  be the

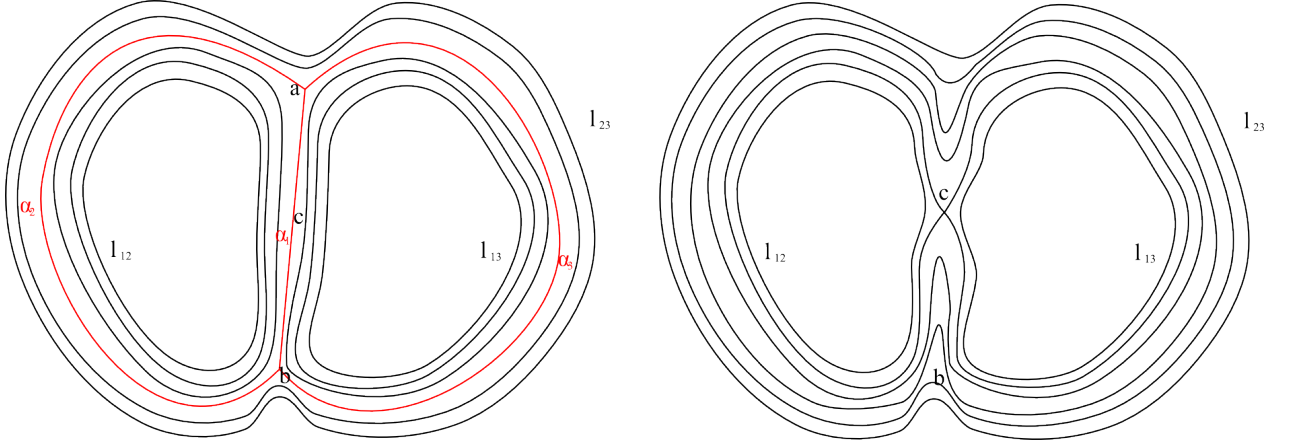


FIGURE 9. Surgery in the proof of Lemma 18

boundary component of the tubular neighbourhood that does not intersect  $\alpha_k$ , ( $k \neq i, j$ ). After a small perturbation in the neighbourhood of  $a$  and  $b$  we can assume that  $l_{ij}$  is smooth and there is a diffeomorphism  $f_{ij}$  from  $(0, 1] \times S^1$  onto the region between  $\alpha_i \cup \alpha_j$  and  $l_{ij}$  with  $|f_{ij}(t \times S^1)| \leq |\alpha_i \cup \alpha_j| + \epsilon$ .

Let  $c$  be the midpoint of  $\alpha_1$ . We perform a straightforward surgery to the curves  $\{f_{ij}(t \times S^1)\}$  depicted on Figure 9. In the neighbourhood of  $c$  we can choose a coordinate chart so that curves in the new foliation are given by  $f(x, y) = 0.5 - x^2 + y^2$ .  $\square$

Now we prove the analogue of Lemmas 13 and 14 for  $m$ -functions.

**Lemma 19.** *For any  $\epsilon > 0$  there exists  $A > 0$ , such that for every disc  $D \subset M$  with  $|D| \leq A$  there exists an  $m$ -map  $f$  from  $D$  with fibers of length less than  $|\partial D| + \epsilon$ .*

*Proof.* As in the proof of Lemma 13, we proceed by induction on  $n = \lceil \frac{|\partial D|}{\sqrt{|A|}} \rceil$ . However, the inductive assumption is now different. We would like to show that for every  $\epsilon > 0$  a subdisc  $D \subset M$  admits an  $m$ -map with fibers of length

$$\leq |\partial D| + (4 + 2\sqrt{2})\sqrt{|D|} + \epsilon'$$

Assume the Lemma to be true for all subdiscs with  $\lceil \frac{|\partial D|}{\sqrt{|A|}} \rceil < n$ .

We take small tubular neighbourhood of  $\partial D$  in  $D$  and foliate it by closed curves of length  $\leq |\partial D| + \epsilon'$ . We subdivide the innermost curve  $\gamma$  into 4 arcs of equal length. By Besicovitch Lemma we can find an arc  $\alpha$  of length  $\leq \sqrt{|D|}$  connecting two opposite subarcs of  $\gamma$ .



Let  $N$  be the neighbourhood of  $\gamma \cup \alpha$  as in Lemma 18. Note that  $\partial N$  has 3 connected components: one of them is  $\partial D$  and denote the other two by  $C_1$  and  $C_2$ , bounding subdiscs  $D_1$  and  $D_2$  respectively. Assume that  $D_1$  is a disc of smaller area, hence  $|D_1| < \frac{1}{2}|D|$ .

$N$  admits an  $m$ -map  $f_0 : N \rightarrow [0, 2]$  with fibers of length  $|f^{-1}(x)| \leq |\partial D| + 2\sqrt{|D|} + \epsilon'$  for  $x \in [0, 1]$  and  $|f^{-1}(x)| \leq |\partial D_2| + \epsilon'$  for  $x \in (1, 2]$ .

Since  $|\partial D_1| \leq (n-1)\sqrt{A}$  by inductive assumption  $D_1$  admits an  $m$ -map  $f_1$  with fibers of length  $< |\partial D_1| + \frac{4+2\sqrt{2}}{\sqrt{2}}\sqrt{|D|} + \epsilon'$ . After appropriately scaling  $f_1$  on a small neighbourhood of  $C_1$  and multiplying by  $-1$  if necessary we can assume that  $f_1(C)$  is the minimum point of  $f_1$ . Furthermore, we scale and shift  $f_1$  so that  $f_1(C_1) = 1$  and  $f_1(D_1)$  is contained between 1 and 1.5. We now extend the  $m$ -map to  $N \cup D_1$  with fibers of length

$$\begin{aligned} &\leq |\partial D_1| + |\partial D_2| + \frac{4+2\sqrt{2}}{\sqrt{2}}\sqrt{|D|} + \epsilon' \\ &\leq |\partial D| + (4+2\sqrt{2})\sqrt{|D|} + \epsilon' \end{aligned}$$

By inductive assumption  $D_2$  admits an  $m$ -function  $f_2$  with fibers of length  $\leq |\partial D| + (4+2\sqrt{2})\sqrt{|D|} + \epsilon'$ . We modify  $f_2$  so that it takes on its minimum at  $f_2(C_2) = 2$ . We can now extend  $f$  to the whole of  $D$ .

Setting  $A < \epsilon^2/100$  and  $\epsilon' < \epsilon/10$  we obtain the desired result.  $\square$

**Lemma 20.** *For any  $\epsilon > 0$  there exists  $A > 0$ , such that for every submanifold with boundary  $M_k \subset M_p$ , with  $|M_k| \leq A$  there exists an  $m$ -map  $f$  from  $M_k$  with fibers of length  $\leq 2|\partial M_k| + 4(k-1)\sqrt{A} + \epsilon$ .*

*Proof.* As in the proof of Lemma 14 we connect all components of  $\partial M_k$  with  $(k-1)$  closed curves  $\gamma_i$  of total length  $\leq 2(k-1)\sqrt{|M_k|}$  and denote the union of  $\gamma_i$ s and  $\partial M_k$  by  $C$ . Denote the normal  $\delta$ -neighbourhood of  $C$  in  $M_k$  by  $N$ . After a small perturbation we can assume that the boundary of  $N$  is smooth.

The complement  $D$  of  $N$  in  $M_k$  is a disc and so by Lemma 19 it admits an  $m$ -map with fibers of length  $\leq |\partial M_k| + 4(k-1)\sqrt{A} + \epsilon$ . Next we proceed as in the proof of Lemma 14 extending the domain of the  $m$ -function over connected components of  $\partial M_k$  one by one.

Let  $\{\beta_i\}_{i=1}^k$  be a collection of nested simple closed curves in  $N$  and  $N_i$  denote the subset of  $N$  between  $\beta_i$  and  $\beta_{i+1}$ . We require that  $|\beta_i| \leq |\partial M_k| + 4(k-1)\sqrt{A} + \epsilon$ ,  $\beta_1 = \partial D$  and for each  $i$   $N_i$  contains exactly one connected component of  $\partial M_k$ . We can then apply Lemma 18 to extend the  $m$ -function from  $D$  to  $N_1$  and inductively to the whole of  $M_k$ .

By appropriately scaling and shifting functions obtained at each step we ensure that the desired bound on the lengths of fibers is maintained.  $\square$

Now we are ready to prove Theorem 17. The proof is similar to that of Theorem 10. Again we proceed by induction on  $\lceil \log_{4/3}(\frac{|M_p|}{A}) \rceil$ .

Fix  $\epsilon > 0$  and let  $A$  be as in Lemma 20. Let  $N = \lceil \log_{4/3}(\frac{|M_p|}{A}) \rceil$ . We claim that for every  $M_k \subset M_p$  with  $(\frac{3}{4})^{n+1}|M_p| < |M_k| \leq (\frac{3}{4})^n|M_p|$  there exists an  $m$ -map with fibers of length

$$(2) \quad \leq |\partial M_k| + 4(N - n + k)\sqrt{A} + \sum_{i=n}^{N-1} 2c(r)\left(\frac{3}{4}\right)^{i/2} \sqrt{|M_p|}$$

Choose  $\gamma \in S(M_k, r)$  of length  $\leq c(r) + \epsilon'$ .

**Case 1.**  $\gamma$  is a simple closed curve.

Let  $N_1$  and  $N_2$  denote the two components of  $M_p \setminus \gamma$ . In this case there is an  $m$ -function  $f_1$  from  $N_1$  onto  $[a, b]$  and an  $m$ -function  $f_2$  onto  $[b, c]$ . Hence, we obtain an  $m$ -function  $f : M_p \rightarrow [a, c]$  satisfying the same bound on the length of the fibers. Using the inductive assumption we calculate that this length bound is exactly what we want.

**Case 2.**  $\gamma$  is a collection of arcs with endpoints on the boundary of  $\partial M_p$ .

As in the proof Theorem 10 we add a small collar around boundary components of  $M_p$  that intersect  $\gamma$ . We apply the inductive assumption to  $B \subset M_p$  and extend this map to regions separated from  $B$  by “horseshoes”. Then we extend the map to collars whose intersection with  $\partial B$  is a connected arc. We iterate this procedure until we extended the map to all of  $M_p$ . By appropriately scaling the  $m$ -functions obtained for each region we ensure the correct bound on the lengths of fibers.

## REFERENCES

- [1] J. C. Alvarez Paiva, F. Balacheff, K. Tzanev, Isosystolic inequalities for optical hypersurfaces, preprint, math arXiv:1308.5522
- [2] F. Balacheff, Measurements of Riemannian two-disks and two-spheres, preprint, math arXiv:1401.7184
- [3] F. Balacheff, S. Sabourau, Diastolic and isoperimetric inequalities on surfaces, *Ann. Sci. Ecole Norm. Sup.* 43 (2010) 579-605.
- [4] C. B. Croke, Area and the length of the shortest closed geodesic, *J. Differential Geom.* 27 (1988), no. 1, 121.
- [5] G. R. Chambers, R. Rotman, Contracting loops on a Riemannian 2-surface, preprint, math arXiv:1311.2995
- [6] L. Guth, Lipschitz maps from surfaces. *Geom. Funct. Anal.* 15 (2005), No 5, 1052 - 1099.
- [7] Y. Liokumovich, A. Nabutovsky, R. Rotman, Contracting the boundary of a Riemannian 2-disc, preprint, math arXiv:1205.5474

- [8] A. Nabutovsky, Contracting the boundary of a Riemannian 2-disc, Dubrovnik VII - Geometric Topology Conference in Inter-University Centre, Dubrovnik, June 26 - July 3, 2011
- [9] P. Papasoglu, Cheeger constants of surfaces and isoperimetric inequalities, *Trans. Amer. Math. Soc.* 361 (2009), no. 10, 5139-5162.
- [10] R. Rotman, The length of a shortest closed geodesic and the area of a 2-dimensional sphere, *Proc. Amer. Math. Soc.* 134 (2006), no. 10, 3041-3047
- [11] S. Sabourau, Filling radius and short closed geodesics of the 2-sphere, *Bull. Soc. Math. France* 132 (2004), 105-136.